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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Remark: Please let me know if there are typos or mistakes.

Question 1

Each of the following functions are defined on $[-\pi, \pi]$. Sketch the 2π-periodic extension, find the corresponding Fourier expansion, and discuss the pointwise convergence.

(a) $f_1(x) = \begin{cases} x, & \text{if } x \in [0, \pi] \\ 0, & \text{if } x \in [-\pi, 0) \end{cases}$

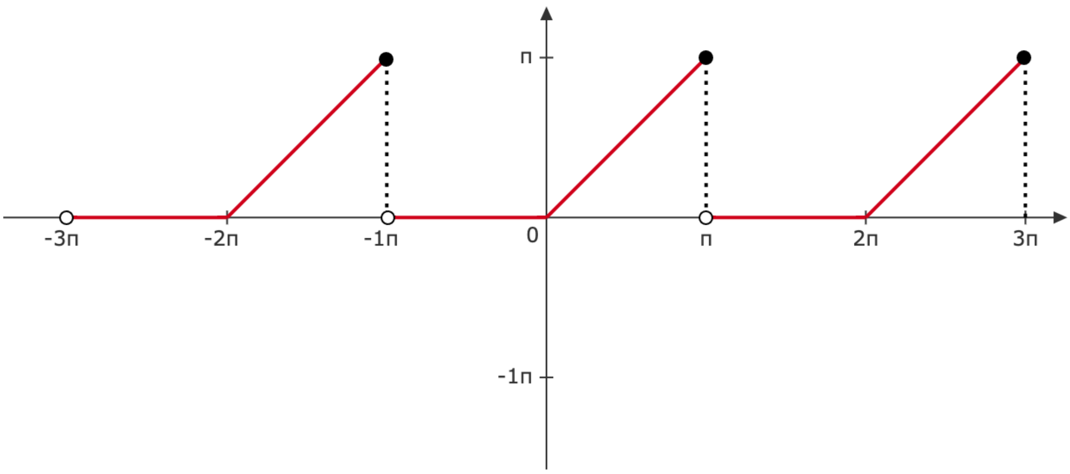
(b) $f_2(x) = \begin{cases} -1, & \text{if } x \in [0, \pi] \\ +1, & \text{if } x \in [-\pi, 0) \end{cases}$

(c) $f_3(x) = e^x$

Solution:

(a) $f_1(x) = \begin{cases} x, & \text{if } x \in [0, \pi] \\ 0, & \text{if } x \in [-\pi, 0) \end{cases}$

Sketch[1] of the 2π-extension:



Remark: Those labels are π, not n.

[1]Sketched by using https://www.mathcha.io/editor

Fourier expansion:

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)\,dx = 0 + \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left(\frac{\cos(\pi n)-1}{n^2}\right) = \frac{(-1)^n-1}{\pi n^2} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx = \frac{1}{\pi}\left(\frac{\sin(n\pi)-n\pi\cos(n\pi)}{n^2}\right) = \frac{(-1)^{n+1}}{n}
\end{aligned}
$$

hence

$$f_1(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos(nx) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

Pointwise convergence:
For $x \in (-\pi, 0)$: $|f_1(x) - f_1(y)| = 0 \le |x-y|$ for all $y \in (-\pi, 0)$;
For $x \in (0, \pi)$: $|f_1(x) - f_1(y)| = |x-y|$ for all $y \in (0, \pi)$;
For $x = 0$: $|f_1(x) - f_1(y)| = |f_1(y)| \le |y| = |x-y|$ (since $0 \le |y|$) for all $y \in (-\pi, \pi)$
Then f_1 is Lipschitz continuous at every $x \in (-\pi, \pi)$.
By Theorem 1.5, the partial sum sequence $\{S_n f_1(x)\}$ converges to $f_1(x)$ pointwisely for each $x \in (-\pi, \pi)$.

By observation, we see $f_1(\pi^+) = \lim\limits_{x\to\pi^+} f_1(x) = 0$ and $f_1(\pi^-) = \lim\limits_{x\to\pi^-} f_1(x) = \pi$. Then check:
Pick $\delta = \pi$, then

$$|f_1(x) - f_1(\pi^+)| = 0 \le |x-\pi| \text{ for all } x,\ 0 < x-\pi < \delta$$

and

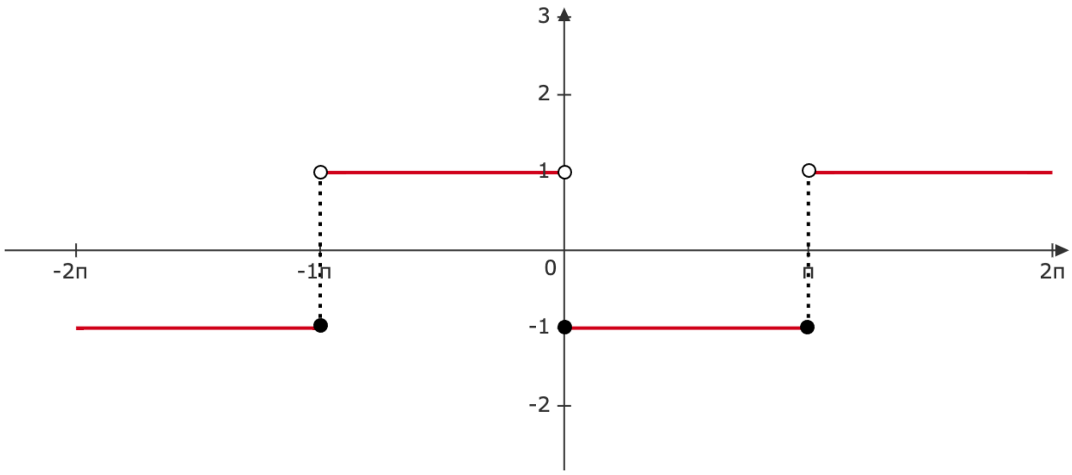
$$|f_1(\pi^-) - f_1(x)| = |\pi - x| \text{ for all } x,\ 0 < \pi - x < \delta$$

Then $S_n f_1(\pi) \to \pi/2$ as $n \to \infty$. Similarly, $S_n f_1(-\pi) \to \pi/2$ as $n \to \infty$.

$\square$

(b) $f_2(x) = \begin{cases} -1, & \text{if } x \in [0, \pi] \\ +1, & \text{if } x \in [-\pi, 0) \end{cases}$

Sketch of the 2π-extension:



Fourier expansion:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(x)\, dx = 0$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_2(x) \cos nx\, dx = 0$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_2(x) \sin nx\, dx = \frac{2}{\pi} \frac{(-1)^n - 1}{n}$$

hence

$$f_2(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^n - 1}{n} \sin(nx)$$

Pointwise convergence:

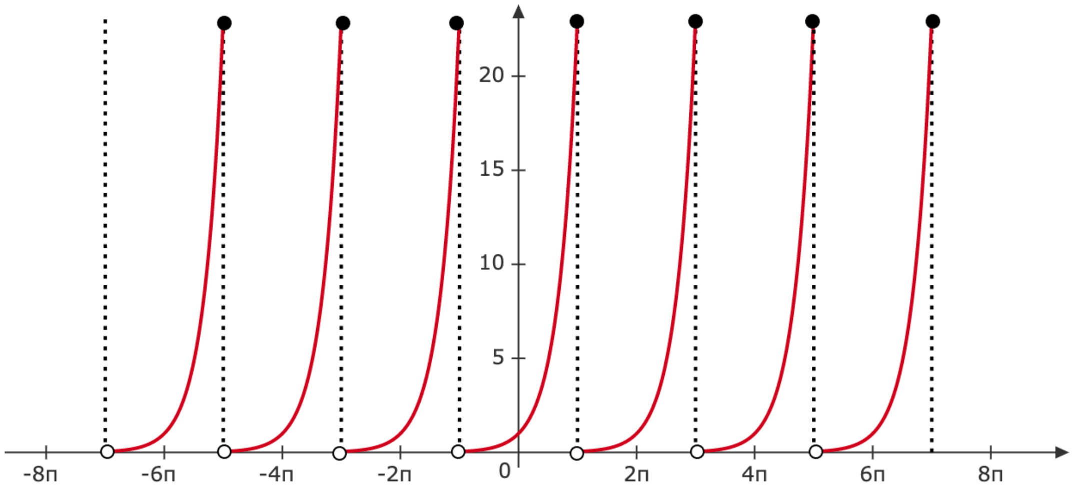
For $x \in (-\pi, 0)$, we have $|f_2(x) - f_2(y)| = 0 \le |x - y|$ for all $y \in (-\pi, 0)$;
For $x \in (0, \pi)$, we have $|f_2(x) - f_2(y)| = 0 \le |x - y|$ for all $y \in (0, \pi)$;
Then f_2 is Lipschitz continuous at $x \in (-\pi, 0) \cup (0, \pi)$ (as f_2 is not continuous at 0, then f_2 is not Lipschitz continuous at 0)
By Theorem 1.5, the partial sum sequence $\{S_n f_2(x)\}$ converges to $f_2(x)$ pointwisely for each $x \in (-\pi, 0) \cup (0, \pi)$.

By observation, $f_2(\pi^+) = \lim_{x \to \pi^+} f_2(x) = 1$, $f_2(\pi^-) = \lim_{x \to \pi^-} f_2(x) = -1$. Pick $\delta = \pi$, then we have $|f_2(x) - f_2(\pi^+)| = 0 \le |x - \pi|$ for all x, $0 < x - \pi < \delta$; $|f_2(x) - f_2(\pi^-)| = 0 \le |x - \pi|$ for all x, $0 < \pi - x < \delta$. So, by theorem 1.6 $S_n f_2(\pi) \to 0$ as $n \to \infty$. Similarly, $S_n f_2(-\pi)$ and $S_n f_2(0)$ both converges to 0 as $n \to \infty$.

$\square$

(c) $f_3(x) = e^x$

Sketch of the 2π-extension:



Fourier expansion:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx \\ &= \frac{e^{\pi}-e^{-\pi}}{2\pi} \\ &= \frac{1}{\pi}\sinh(\pi) \end{aligned}$$

where

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Next, for convenience, we apply

$$\int e^{ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}e^{ax}\cos(bx) + \frac{b}{a^2+b^2}e^{ax}\sin(bx) + C$$

then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{1+n^2}e^x\cos(nx) + \frac{n}{1+n^2}e^x\sin(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[\frac{1}{1+n^2}e^{\pi}\cos(n\pi) - \frac{1}{1+n^2}e^{-\pi}\cos(-n\pi)\right] \\ &= \frac{(-1)^n}{\pi}\frac{2\sinh(\pi)}{1+n^2} \end{aligned}$$

Similarly, we apply

$$\int e^{ax}\sin(bx)\,dx = \frac{a}{a^2+b^2}e^{ax}\sin(bx) - \frac{b}{a^2+b^2}e^{ax}\cos(bx) + C$$

then

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{1+n^2}e^x\sin(nx) - \frac{n}{1+n^2}e^x\cos(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[-\frac{n}{1+n^2}e^{\pi}\cos(n\pi) + \frac{n}{1+n^2}e^{-\pi}\cos(-n\pi)\right] \\ &= \frac{n(-1)^{n+1}}{\pi}\frac{2\sinh(\pi)}{1+n^2} \end{aligned}$$

hence, its Fourier expansion is given by

$$f_3(x) = e^x \sim \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx) - n\sin(nx)}{1+n^2}$$

Pointwise convergence:
For any $x \in (-\pi,\pi)$, we have, by mean value theorem, that $|e^x - e^y| = e^{x_0}|x-y|$ for some $x_0 \in (x,y)$, then $|e^x - e^y| \leq e^{\pi}|x-y|$, for any $y \in (-\pi,\pi)$. Thus, it is Lipschitz continuous at any $x \in (-\pi,\pi)$.

By Theorem 1.5, $S_nf_3(x) \to f_3(x)$ for all $x \in (-\pi,\pi)$.

By observation, $f_3(\pi^+) = \lim_{x\to\pi^+} e^x = e^{-\pi}$ and $f_3(\pi^-) = \lim_{x\to\pi^-} e^x = e^{\pi}$. Similar calculation tells us

$$S_nf_3(\pi) \to \frac{e^{-\pi}+e^{\pi}}{2} = \cosh(\pi)$$

as $n\to\infty$. Similar for $S_nf(-\pi) \to \sinh(\pi)$ as $n\to\infty$.

$\square$

Question 2

Show that the function $f(x) = |x|^\alpha$, $x \in [-\pi, \pi]$ is not Lipschitz continuous at $x = 0$ for any $0 < \alpha < 1$.

Solution:
We argue by contradiction[2].

Suppose $f(x) = |x|^\alpha$ is Lipschitz continuous at $x = 0$, then by definition, there exists $L > 0$ and $\delta > 0$ such that $|f(x) - f(0)| = ||x|^\alpha - |0|^\alpha| = |x|^\alpha < L|x|$ for all $x \in [a, b]$ and $|x| < \delta$. Following the definition, we have $|x|^{\alpha-1} < L$. But δ can be made arbitrarily small, thus, as $\delta \to 0^+$, we have $x \to 0$ and $|x|^{\alpha-1} \to \infty$, since $-1 < \alpha - 1 < 0$. Then we have $\infty < L$, which is impossible. Hence, f is not Lipschitz continuous at $x = 0$.

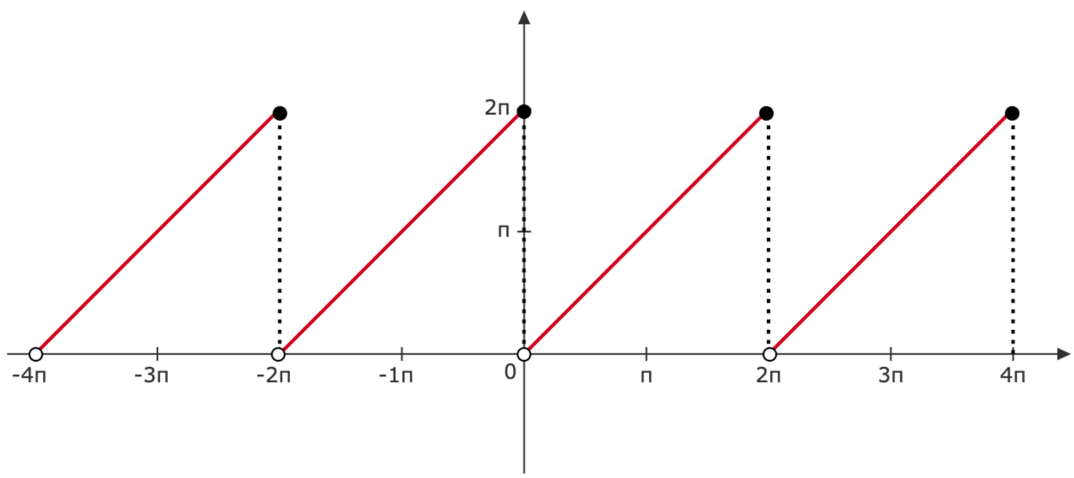
$\square$

Question 3

Consider the function $f(x) = x$ on $(0, 2\pi]$ and its 2π-periodic extension $\tilde{f}(x) = f(x - 2k\pi)$ for $x \in (2k\pi, 2(k+1)\pi]$, for all $k \in \mathbb{Z}$. Sketch $\tilde{f}$, find its Fourier series, and discuss the pointwise convergence. Finally, if the Fourier series converges at the point $x = 0$, what value does it limit to? (Compare with $f(x) = x$ on $[-\pi, \pi]$).

Solution:

Sketch of $\tilde{f}$:



[2]You can also prove it directly.

Fourier series of $\tilde{f}$:

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} x + 2\pi \, dx + \int_0^{\pi} x \, dx\right) = \pi \\
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x + 2\pi)\cos(nx) \, dx + \int_0^{\pi} x\cos(nx) \, dx\right) = 0 \\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x + 2\pi)\sin(nx) \, dx + \int_0^{\pi} x\sin(nx) \, dx\right) = -\frac{2}{n}
\end{aligned}
$$

thus its Fourier series is given by

$$\tilde{f}(x) \sim \pi - \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$$

Pointwise convergence:

On $(0, 2\pi)$, we know that $|f(x) - f(y)| = |x - y|$ for all $y \in (0, 2\pi)$. Thus f is Lipschitz continuous at any $x \in (0, 2\pi)$. By Theorem 1.5, we have $S_n f(x) \to f(x)$ as $n \to \infty$ for any $x \in (0, 2\pi)$.

Consider $x = 0$. $f(0^+) = 0$ and $f(0^-) = 2\pi$. Pick $\delta = \pi$ and consider $0 < x < \delta$, we have

$$|f(x) - f(0^+)| = |x - 0|$$

and for $-\delta < x < 0$, we have

$$|f(x + 2\pi) - f(0^-)| = |x + 2\pi - 2\pi| = |x - 0|$$

hence by theorem 1.6,

$$S_n f(0) \to \pi$$

as $n \to \infty$.

Comparison:

Refer to Prof Wan's hand written lecture notes: Lecture 2 and Lecture 3

$\square$

Question 4

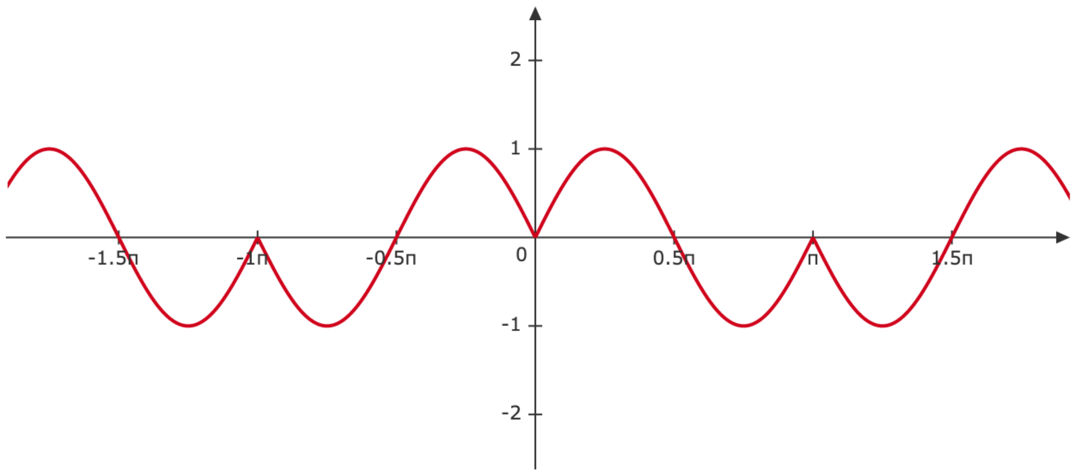
Consider the function $f(x) = \sin(2x)$ on $(0, \pi]$ and extend to an even function $f_1(x)$ on $[-\pi, \pi]$, then further extend f_1 to a 2π-periodic function $\tilde{f}_1$ as usual. Sketch $\tilde{f}_1$. Show that

$$\tilde{f}_1(x) \sim \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{1}{4-(2k+1)^2} \cos((2k+1)x).$$

Discuss the pointwise and uniform convergence. (Compare with $\sin(2x)$ on $[-\pi, \pi]$).

Solution:

Sketch of $\tilde{f}_1$:



Fourier series:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\sin(2x)\, dx + + \int_0^{\pi} \sin(2x)\right) = 0 \\
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(2x)\cos(nx)\, dx + + \int_0^{\pi} \sin(2x)\cos(nx)\right) = \frac{4((-1)^{n+1}+1)}{4-n^2} \\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(2x)\sin(nx)\, dx + + \int_0^{\pi} \sin(2x)\sin(nx)\right) = 0
\end{aligned}$$

note that for $n = 2k$, we have $(-1)^{2k+1} = -1$ implies $a_{2k} = 0$ for all $k \in \mathbb{N}$. Thus, we are left with

$$a_{2k+1} = \frac{8}{4-(2k+1)^2}$$

thus,

$$\tilde{f}_1(x) \sim \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{1}{4-(2k+1)^2} \cos((2k+1)x)$$

Pointwise Convergence:

$f(x)$ satisfies the Lipschitz condition on $\mathbb{R}$:

$$|f(x) - f(y)| = \begin{cases} |\sin(2x) - \sin(2y)| & x, y \in (2k\pi, (2k+1)\pi] \\ |\sin(2y) - \sin(2x)| & x, y \in ((2k+1)\pi, 2k\pi] \end{cases} = 2|\cos(2x_0)||x-y| \leq 2|x-y|$$

for all $k \in \mathbb{Z}$ and for some $x_0 \in (x, y)$ by mean value theorem. Thus f is Lipschitz on $\mathbb{R}$.

By theorem 1.5, $S_n f$ converges to f as $n \to \infty$ pointwisely on $[-\pi, \pi]$, and by theorem 1.7, $S_n f$ converges uniformly to f on $\mathbb{R}$.

Comparison:
The "2π-periodic expansion" of $f(x) = \sin(2x)$ on $[-\pi, \pi]$ is $f(x)$ itself defined on $\mathbb{R}$. So the only coefficient of its Fourier series is $b_2 = \pi$, in other words, its Fourier series is exactly $\sin(2x)$. $\square$